

EXISTENCE OF SPECIAL PRIMARY DECOMPOSITIONS IN MULTIGRADED MODULES

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ABSTRACT. Let $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{\underline{n}}$ be a commutative Noetherian \mathbb{N}^t -graded ring, and $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{\underline{n}}$ be a finitely generated \mathbb{N}^t -graded R -module. We prove that there exists a positive integer k such that for any $\underline{n} \in \mathbb{N}^t$ with $L_{\underline{n}} \neq 0$, there exists a primary decomposition of the zero submodule $O_{\underline{n}}$ of $L_{\underline{n}}$ such that for any $P \in \text{Ass}_{R_0}(L_{\underline{n}})$, the P -primary component Q in that primary decomposition contains $P^k L_{\underline{n}}$. We also give an example which shows that not all primary decompositions of $O_{\underline{n}}$ in $L_{\underline{n}}$ have this property. As an application of our result, we prove that there exists a fixed positive integer l such that the 0^{th} local cohomology $H_I^0(L_{\underline{n}}) = (0 :_{L_{\underline{n}}} I^l)$ for all ideals I of R_0 and for all $\underline{n} \in \mathbb{N}^t$.

1. INTRODUCTION

Let A be a commutative Noetherian ring with unity, and let $N \subsetneq M$ be two finitely generated A -modules. Let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ be an irredundant and minimal primary decomposition of N in M , where Q_i is a P_i -primary submodule of M , i.e., $\text{Ass}_A(M/Q_i) = \{P_i\}$ for all $i = 1, 2, \dots, r$. We call Q_i as a P_i -primary component of N in M . In this case, we have $\text{Ass}_A(M/N) = \{P_1, P_2, \dots, P_r\}$. Note that $\sqrt{\text{Ann}_A(M/Q_i)} = P_i$, and hence there exists some positive integer s_i such that $P_i^{s_i} M \subseteq Q_i$ for each $i = 1, 2, \dots, r$. From now onwards, by a primary decomposition, we always mean an irredundant and minimal primary decomposition unless explicitly stated otherwise.

Through out this article, we denote $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{\underline{n}}$ as a commutative Noetherian \mathbb{N}^t -graded ring with unity (where \mathbb{N} is the collection of all non-negative integers and t is any fixed positive integer) and $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{\underline{n}}$ as a finitely generated \mathbb{N}^t -graded R -module unless explicitly stated otherwise. Set $A = R_{(0, \dots, 0)}$. Note that A is a Noetherian ring, and each $L_{\underline{n}}$ is a finitely generated A -module. We denote $O_{\underline{n}}$ as the zero submodule of $L_{\underline{n}}$ for each $\underline{n} \in \mathbb{N}^t$.

For each $\underline{n} \in \mathbb{N}^t$, fix a primary decomposition

$$(\dagger) \quad O_{\underline{n}} = Q_{\underline{n},1} \cap Q_{\underline{n},2} \cap \cdots \cap Q_{\underline{n},r_{\underline{n}}}$$

of $O_{\underline{n}}$ in $L_{\underline{n}}$, where $Q_{\underline{n},i}$ is a $P_{\underline{n},i}$ -primary component of $O_{\underline{n}}$ in $L_{\underline{n}}$. Then there exists a positive integer $s_i(\underline{n})$ such that $P_{\underline{n},i}^{s_i(\underline{n})} L_{\underline{n}} \subseteq Q_{\underline{n},i}$ for each $\underline{n} \in \mathbb{N}^t$ and $i = 1, 2, \dots, r_{\underline{n}}$. A natural question arises that “can we choose a primary decomposition (\dagger) of each $O_{\underline{n}}$ for which we have $s_i(\underline{n})$ bounded irrespective of i and \underline{n} ?”.

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In this article, we see that there exists some primary decomposition (\dagger) of each $O_{\underline{n}}$ for which we can choose $s_i(\underline{n})$ in such a way that it is bounded (see Theorem 2.4). This confirms a conjecture by Tony J. Puthenpurakal. In an example, we also show that not all primary decompositions of $O_{\underline{n}}$ have this property (see Example 2.5).

Application. Suppose I is an ideal of A , and M is an A -module. Recall that the 0^{th} local cohomology module $H_I^0(M)$ of M with respect to I is defined to be

$$H_I^0(M) := \{x \in M \mid I^n x = 0 \text{ for some } n \in \mathbb{N}\} = \bigcup_{n \in \mathbb{N}} (0 :_M I^n).$$

If M is a Noetherian A -module, then it is easy to see that $H_I^0(M) = (0 :_M I^e)$ for some $e \in \mathbb{N}$, where e depends on M as well as I . If $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{\underline{n}}$ is a finitely generated \mathbb{N}^t -graded R -module, then as an application of our main result on primary decomposition, we prove that there exists a fixed positive integer l such that $H_I^0(L_{\underline{n}}) = (0 :_{L_{\underline{n}}} I^l)$ for all ideals I of R_0 and for all $\underline{n} \in \mathbb{N}^t$ (see Theorem 3.1).

2. MAIN RESULT

To prove our main result (Theorem 2.4), we need some preliminaries. We start with the following lemma.

Lemma 2.1. *Let $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{\underline{n}}$ be a Noetherian \mathbb{N}^t -graded ring, and $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{\underline{n}}$ be a finitely generated \mathbb{N}^t -graded R -module. Set $A = R_{(0, \dots, 0)}$. Let J be an ideal of A . Then there exists a positive integer k such that*

$$J^m L_{\underline{n}} \cap H_J^0(L_{\underline{n}}) = O_{\underline{n}} \quad \forall \underline{n} \in \mathbb{N}^t \text{ and } \forall m \geq k.$$

Proof. Let $I = JR$ be the ideal of R generated by J . Since R is Noetherian and L is a finitely generated R -module, then by Artin-Rees lemma, there exists a positive integer c such that

$$(2.1.1) \quad \begin{aligned} (I^m L) \cap H_I^0(L) &= I^{m-c}((I^c L) \cap H_I^0(L)) \quad \text{for all } m \geq c \\ &\subseteq I^{m-c}(H_I^0(L)) \quad \text{for all } m \geq c. \end{aligned}$$

Now consider the ascending chain of submodules of L :

$$(0 :_L I) \subseteq (0 :_L I^2) \subseteq (0 :_L I^3) \subseteq \dots$$

Since L is a Noetherian R -module, there exists some l such that

$$(2.1.2) \quad (0 :_L I^l) = (0 :_L I^{l+1}) = \dots = H_I^0(L).$$

Set $k := c + l$. Then from (2.1.1) and (2.1.2), for all $m \geq k$, we have

$$(I^m L) \cap H_I^0(L) \subseteq I^{m-c}(0 :_L I^{m-c}) = 0,$$

which gives $J^m L_{\underline{n}} \cap H_J^0(L_{\underline{n}}) = O_{\underline{n}}$ for all $\underline{n} \in \mathbb{N}^t$ and for all $m \geq k$. \square

An immediate corollary is the following.

Corollary 2.2. *Let $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{\underline{n}}$ be a Noetherian \mathbb{N}^t -graded ring, and $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{\underline{n}}$ be a finitely generated \mathbb{N}^t -graded R -module. Set $A = R_{(0, \dots, 0)}$. Then there exists a positive integer k such that for any $P \in \bigcup_{\underline{n} \in \mathbb{N}^t} \text{Ass}_A(L_{\underline{n}})$, we have*

$$P^k L_{\underline{n}} \cap H_P^0(L_{\underline{n}}) = O_{\underline{n}} \quad \forall \underline{n} \in \mathbb{N}^t.$$

Proof. By [4, Lemma 3.2], we may assume that

$$\bigcup_{\underline{n} \in \mathbb{N}^t} \text{Ass}_A(L_{\underline{n}}) = \{P_1, P_2, \dots, P_l\}.$$

From Lemma 2.1, for each P_i ($1 \leq i \leq l$), there exists some k_i such that

$$P_i^m L_{\underline{n}} \cap H_{P_i}^0(L_{\underline{n}}) = O_{\underline{n}} \quad \forall \underline{n} \in \mathbb{N}^t \text{ and } \forall m \geq k_i.$$

Now the corollary follows by taking $k := \max\{k_i : 1 \leq i \leq l\}$. \square

Let us recall the following result from [3, Theorem 1.1].

Theorem 2.3. *Let A be a Noetherian ring, and $N \subsetneq M$ be two finitely generated A -modules such that $\text{Ass}_A(M/N) = \{P_1, \dots, P_r\}$. Let Q_i be a P_i -primary component of N in M for each $1 \leq i \leq r$. Then $N = Q_1 \cap \dots \cap Q_r$, which is necessarily an irredundant and minimal primary decomposition of N in M .*

We are now in a position to prove our main result.

Theorem 2.4. *Let $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{\underline{n}}$ be a Noetherian \mathbb{N}^t -graded ring, and $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{\underline{n}}$ be a finitely generated \mathbb{N}^t -graded R -module. Set $A = R_{(0, \dots, 0)}$. Then there exists a positive integer k such that for each $\underline{n} \in \mathbb{N}^t$ with $L_{\underline{n}} \neq 0$, there exists a primary decomposition of the zero submodule $O_{\underline{n}}$ of $L_{\underline{n}}$:*

$$O_{\underline{n}} = Q_{\underline{n},1} \cap Q_{\underline{n},2} \cap \dots \cap Q_{\underline{n},r_{\underline{n}}},$$

where $Q_{\underline{n},i}$ is a $P_{\underline{n},i}$ -primary component of $O_{\underline{n}}$ in $L_{\underline{n}}$ satisfying

$$P_{\underline{n},i}^k L_{\underline{n}} \subseteq Q_{\underline{n},i} \quad \forall \underline{n} \in \mathbb{N}^t \text{ and } \forall i = 1, 2, \dots, r_{\underline{n}}.$$

Proof. Let k be as in Corollary 2.2. Fix $\underline{n} \in \mathbb{N}^t$ such that $L_{\underline{n}} \neq 0$. By Theorem 2.3, it is enough to prove that for each $P \in \text{Ass}_A(L_{\underline{n}})$, there exists a P -primary component Q of $O_{\underline{n}}$ in $L_{\underline{n}}$ such that $P^k L_{\underline{n}} \subseteq Q$. So fix $P \in \text{Ass}_A(L_{\underline{n}})$. From Corollary 2.2, we have

$$(2.4.1) \quad P^k L_{\underline{n}} \cap H_P^0(L_{\underline{n}}) = O_{\underline{n}}.$$

It is easy to see that $P^k L_{\underline{n}} \neq L_{\underline{n}}$ (for instance by localization at P). If $H_P^0(L_{\underline{n}}) = L_{\underline{n}}$, then (2.4.1) gives $P^k L_{\underline{n}} = O_{\underline{n}}$, hence any P -primary component of $O_{\underline{n}}$ in $L_{\underline{n}}$ contains $P^k L_{\underline{n}}$, and hence we are through. Therefore we may as well assume that $H_P^0(L_{\underline{n}}) \neq L_{\underline{n}}$. Now we fix irredundant and minimal primary decompositions

$$(2.4.2) \quad P^k L_{\underline{n}} = Q_1 \cap \dots \cap Q_u \quad \text{and} \quad H_P^0(L_{\underline{n}}) = Q_{u+1} \cap \dots \cap Q_v$$

of $P^k L_{\underline{n}}$ and $H_P^0(L_{\underline{n}})$ in $L_{\underline{n}}$ respectively.

We claim that $P \notin \text{Ass}_A(L_{\underline{n}}/H_P^0(L_{\underline{n}}))$. Otherwise $P = \text{Ann}_A(\overline{m})$ for some $m \in L_{\underline{n}} \setminus H_P^0(L_{\underline{n}})$. But then $Pm \subseteq H_P^0(L_{\underline{n}})$ implies that $P^j(Pm) = 0$ in $L_{\underline{n}}$ for some $j \in \mathbb{N}$, i.e., $m \in H_P^0(L_{\underline{n}})$, which is a contradiction. Therefore $P \notin \text{Ass}_A(L_{\underline{n}}/H_P^0(L_{\underline{n}}))$. From (2.4.1), it can be noted that

$$P \in \text{Ass}_A(L_{\underline{n}}) \subseteq \text{Ass}_A(L_{\underline{n}}/P^k L_{\underline{n}}) \cup \text{Ass}_A(L_{\underline{n}}/H_P^0(L_{\underline{n}})).$$

Since $P \notin \text{Ass}_A(L_{\underline{n}}/H_P^0(L_{\underline{n}}))$, we have $P \in \text{Ass}_A(L_{\underline{n}}/P^k L_{\underline{n}})$, and hence without loss of generality, we may assume that $Q := Q_1$ is a P -primary submodule of $L_{\underline{n}}$. Now considering (2.4.1) and (2.4.2), we have a primary decomposition

$$O_{\underline{n}} = Q_1 \cap \dots \cap Q_u \cap Q_{u+1} \cap \dots \cap Q_v,$$

which need not be irredundant or minimal, but we get the desired P -primary component $Q = Q_1$ of $O_{\underline{n}}$ in $L_{\underline{n}}$ such that $P^k L_{\underline{n}} \subseteq Q$. \square

Now we give an example not all primary decompositions have $s_i(\underline{n})$ bounded.

Example 2.5. [2, page 299] Let K be a field. Set $A := K[X, Y]$ as the polynomial ring in two variables X, Y over K . Let $I = (X^2, XY)$. Set $R := A[Z]$ and $L := A/I[Z]$ with natural \mathbb{N} -grading structures, where Z is an indeterminate. Here for each $n \in \mathbb{N}$, $L_n = (A/I)Z^n \cong A/I$ as A -modules. For each O_n (zero submodule of L_n), fix the primary decomposition

$$O_n = (x^2, xy) = (x) \cap (x^2, xy, y^{n+1}) = Q_{n,1} \cap Q_{n,2} \text{ (say),}$$

where x, y are the images of X, Y in A/I respectively. Here $Q_{n,1} = (x)$ is a (X) -primary and $Q_{n,2} = (x^2, xy, y^{n+1})$ is a (X, Y) -primary submodules of L_n . For $n \geq 1$, the minimal $s_2(n)$ we can choose so that

$$(X, Y)^{s_2(n)} L_n = (x, y)^{s_2(n)} \subseteq Q_{n,2} = (x^2, xy, y^{n+1})$$

is $(n+1)$, which is unbounded.

3. AN APPLICATION

As a consequence of the Theorem 2.4, we prove the following result (Theorem 3.1) which says that for any ideal I of A and for any $\underline{n} \in \mathbb{N}^t$, the 0^{th} local cohomology module $H_I^0(L_{\underline{n}}) = (0 :_{L_{\underline{n}}} I^k)$, where k is a fixed positive integer as occurring in Theorem 2.4.

Theorem 3.1. *Let $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{\underline{n}}$ be a Noetherian \mathbb{N}^t -graded ring, and $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{\underline{n}}$ be a finitely generated \mathbb{N}^t -graded R -module. Set $A = R_{(0, \dots, 0)}$. Then there exists a positive integer l such that for any ideal I of A , we have*

$$H_I^0(L_{\underline{n}}) = (0 :_{L_{\underline{n}}} I^l) \quad \forall \underline{n} \in \mathbb{N}^t.$$

Proof. By virtue of the Theorem 2.4, there exists a positive integer l such that for each $\underline{n} \in \mathbb{N}^t$ with $L_{\underline{n}} \neq 0$, we fix a primary decomposition of the zero submodule $O_{\underline{n}}$ of $L_{\underline{n}}$:

$$(3.1.1) \quad O_{\underline{n}} = Q_{\underline{n},1} \cap Q_{\underline{n},2} \cap \dots \cap Q_{\underline{n},r_{\underline{n}}},$$

where $Q_{\underline{n},i}$ is a $P_{\underline{n},i}$ -primary component of $O_{\underline{n}}$ in $L_{\underline{n}}$ satisfying

$$(3.1.2) \quad P_{\underline{n},i}^l L_{\underline{n}} \subseteq Q_{\underline{n},i} \quad \forall \underline{n} \in \mathbb{N}^t \text{ and } \forall i = 1, 2, \dots, r_{\underline{n}}.$$

We claim that the theorem holds true for this l .

Let I be an arbitrary ideal of A . Fix an arbitrary $\underline{n} \in \mathbb{N}^t$. If $L_{\underline{n}} = 0$, then there is nothing to prove. So we may as well assume that $L_{\underline{n}} \neq 0$. By [1, Proposition 3.13 a.], from (3.1.1), we have

$$(3.1.3) \quad H_I^0(L_{\underline{n}}) = \bigcap \{Q_{\underline{n},i} \mid I \not\subseteq P_{\underline{n},i}, 1 \leq i \leq r_{\underline{n}}\}.$$

Then by using (3.1.3), (3.1.2) and (3.1.1) serially, we have

$$\begin{aligned} & (I^l L_{\underline{n}}) \cap H_I^0(L_{\underline{n}}) \\ & \subseteq \left(\bigcap \{P_{\underline{n},i}^l L_{\underline{n}} \mid I \subseteq P_{\underline{n},i}, 1 \leq i \leq r_{\underline{n}}\} \right) \cap \left(\bigcap \{Q_{\underline{n},i} \mid I \not\subseteq P_{\underline{n},i}, 1 \leq i \leq r_{\underline{n}}\} \right) \\ & \subseteq \left(\bigcap \{Q_{\underline{n},i} \mid I \subseteq P_{\underline{n},i}, 1 \leq i \leq r_{\underline{n}}\} \right) \cap \left(\bigcap \{Q_{\underline{n},i} \mid I \not\subseteq P_{\underline{n},i}, 1 \leq i \leq r_{\underline{n}}\} \right) \\ & = Q_{\underline{n},1} \cap Q_{\underline{n},2} \cap \dots \cap Q_{\underline{n},r_{\underline{n}}} = O_{\underline{n}}. \end{aligned}$$

Thus for any ideal I of A , we have

$$(3.1.4) \quad (I^l L_{\underline{n}}) \cap H_I^0(L_{\underline{n}}) = 0 \quad \forall \underline{n} \in \mathbb{N}^t.$$

Let $x \in H_I^0(L_{\underline{n}})$. Then

$$I^l x \subseteq (I^l L_{\underline{n}}) \cap H_I^0(L_{\underline{n}}) = 0,$$

and hence $x \in (0 :_{L_{\underline{n}}} I^l)$. Thus for any ideal I of A , we have

$$H_I^0(L_{\underline{n}}) = (0 :_{L_{\underline{n}}} I^l) \quad \forall \underline{n} \in \mathbb{N}^t,$$

which completes the proof of the theorem. \square

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